The latter shows that the critical values of the wave number a_* become larger as α decreases, which means that perturbations of smaller scale can cause instability, which itself is due to reduced thickness of the region of unstable stratification.

Heterogeneous reaction or loss of reagent proportional to the concentration at the lower boundary, as represented by (1.8c) and (3.2c), will have relatively little effect on the convective stability. The family of $R_*(\Psi)$ curves for values of the Sherwood number between 0 and ∞ lies in the range between the $R_*(\Psi)$ curves corresponding to the boundary conditions of (1.8a), (3.2a) and (1.8b), (3.2b) for α = 0.

NOTATION

v, infiltration rate; C, C_0, $C^{(0)}$, current reagent concentration, equilibrium value, and dimensional value at upper boundary; p, convective correction to pressure; t, time; g, acceleration due to gravity; e, vertical unit vector; $\beta = (1/\rho_0)(\partial\rho/\partial C)_{T,p}$, coefficient relating density to concentration; K, permeability; m, porosity; ν, D, kinematic viscosity and diffusion constant; k, rate constant of homogeneous reaction; σ, rate constant for heterogeneous reaction and mass transfer; w(z), η(z), amplitudes of normal velocity and concentration perturbations; a_1, a_2, wave numbers for perturbations along x and y axes, $a_1^2 + a_2^2 = a^2$; λ, perturbation decrement.

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THREE-DIMENSIONAL POTENTIALS FOR THE TELEGRAPHERS' EQUATION AND THEIR APPLICATION TO BOUNDARY-VALUE HEAT-CONDUCTION PROBLEMS

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Three-dimensional potentials for the telegraphers' equation are introduced and used to reduce boundary-value heat-conduction problems to integrodifferential equations of the second kind.

In recent years the hyperbolic heat-conduction equation has been used to solve various kinds of heat-conduction and thermoelasticity problems [1, 2]. Therefore, it has become necessary to create the mathematical apparatus for solving direct and inverse heat-conduction problems based on the hyperbolic equation. With this in mind we generalize the potential method to the case of the telegraphers' equation.

Three-Dimensional Potentials for the Telegraphers' Equation. We consider the homogeneous telegraphers' equation with constant coefficients

$$-\frac{1}{c^2}\cdot\frac{\partial^2 u(t,\ M_0)}{\partial t^2}-\frac{1}{a}\cdot\frac{\partial u(t,\ M_0)}{\partial t}+du(t,\ M_0)+\Delta u=0 \tag{1}$$

and zero initial conditions

$$u(0,\ M_0)=\partial u(0,\ M_0)/\partial t=0 \tag{2}$$

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in an arbitrary simply connected domain $D(M_0 \in \bar{D})$ bounded by a smooth surface $\Gamma = \bar{D} - D$. In the following discussion we denote Laplace transforms of various functions by the corresponding capital letters. Taking account of (2) the Laplace transform of Eq. (1) becomes

$$\Delta U(p, M_0) - \left(\frac{p^2}{c^2} + \frac{p}{a} - d\right) U(p, M_0) = 0. \tag{3}$$

It is known [3] that in the domain of ellipticity of Eq. (3) three types of potentials can be introduced: a single-layer potential Φ_1, a double-layer potential Φ_2, and a volume potential Φ_3. In the three-dimensional case they have the following form [3]:

$$\Phi_1(p, M_0) = \iint_\Gamma \Psi_1(p, M) \frac{\exp(-rs/c)}{4\pi r} dS, \tag{4}$$

$$\Phi_2(p, M_0) = -\iint_\Gamma \frac{1}{4\pi r} \cdot \frac{\partial r}{\partial n} \Psi_2(p, M) \left[\frac{1}{r} + s/c\right] \exp(-rs/c)\, dS, \tag{5}$$

$$\Phi_3(p, M_0) = \iiint \Psi_3(p, M) \frac{\exp(-rs/c)}{4\pi r} dV, \tag{6}$$

where $s = \sqrt{(p + c^2/2a)^2 - b^2}$, $b = c^2\sqrt{1 + 4a^2 d/c^2}/2a$, n is the normal directed into domain D from point M on the surface Γ, $r = |MM_0|$ is the distance from point M on the domain boundary to point $M_0 \in \bar{D}$, and the arbitrary continuous functions $\Psi_k(p, M)$ are the source densities for the corresponding potentials and depend on p as on a parameter. In a volume potential the integration is performed either over domain D or over the domain $R_3 - D$ external to it, depending on the problem under consideration. In passing through the surface Γ the potential (5) has a discontinuity; potential (4) remains continuous, but its normal derivative has a discontinuity [3]

$$\frac{\partial \Phi_1^{\pm}(p, M_0)}{\partial n_0} = \mp \frac{\Psi_1(p, M_0)}{2} + \frac{\partial \Phi_1(p, M_0)}{\partial n_0}, \quad M_0 \in \Gamma, \tag{4a}$$

$$\Phi_2^{\pm}(p, M_0) = \pm \frac{\Psi_2(p, M_0)}{2} + \Phi_2(p, M_0), \quad M_0 \in \Gamma, \tag{5a}$$

where the plus and minus signs indicate the limiting values of the corresponding functions as point M_0 approaches the boundary Γ from the inside and from the outside; n_0 is the inward normal to the boundary Γ at point M_0. The volume potential (6) is a twice continuously differentiable function in the domain of integration and satisfies the equation

$$\Delta\Phi_3 - \left(\frac{p^2}{c^2} + \frac{p}{a} - d\right) \Phi_3 = -\Psi_3(p, M_0). \tag{6a}$$

Outside the domain of integration the potential (6) satisfies Eq. (3).

We take the inverse transforms of Eqs. (4)-(6), using analytic continuation of the transforms [9], the convolution theorem, the known inverse transform of F(s) [4], and the principle of causality. The single- and double-layer potentials go over respectively into the functions $\varphi_m(t, M_0)$ $(m = 1, 2)$:

$$\varphi_m(t, M_0) = \iint_\Gamma E\left(t - \frac{r}{c}\right) \int_0^t \psi_m(\tau, M) f_m(t-\tau, M, M_0)\, d\tau dS. \tag{7}$$

Here the $\psi_m(t, M)$ are arbitrary functions; f_1 is the solution for a unit instantaneous source and is equal to [5]

$$f_1(t, M, M_0) = \exp(-c^2 t/2a) \left\{\delta(t - r/c)/4\pi r + \frac{b}{4\pi c} E\left(t - \frac{r}{c}\right) \frac{I_1(b\sqrt{t^2 - r^2/c^2})}{\sqrt{t^2 - r^2/c^2}}\right\}, \tag{8}$$

and f_2 is the solution for a unit instantaneous dipole

$$f_2(t, M, M_0) = -\frac{1}{4\pi} \frac{\partial r}{\partial n} \exp(-c^2 t/2a) \left\{\frac{\delta(t - r/c)}{r^2} + \frac{\delta'\left(t - \frac{r}{c}\right)}{rc} + E\left(t - \frac{r}{c}\right) \frac{rb}{c^3} \times \right. \tag{8a}$$

$$\times\left[-2\frac{I_1(b\sqrt{t^2-r^2/c^2})}{\sqrt{t^2-r^2/c^2}}+bI_0(b\sqrt{t^2-r^2/c^2})/\sqrt{t^2-r^2/c^2}\right]\Bigg\}. \quad (8a)$$

Using the properties of the Dirac delta function we write the final expression for the functions φ_m (m = 1, 2) in the following form:

$$\varphi_1(t, M_0)=\iint_\Gamma \frac{\psi_1(t-r/c, M)}{4\pi r} E\left(t-\frac{r}{c}\right)\exp(-rc/2a)\,dS+$$

$$+\frac{b}{4\pi c}\iint_\Gamma E\left(t-\frac{r}{c}\right)\int_0^{t-r/c}\psi_1(\tau, M)\exp[-c^2(t-\tau)/2a]\,\frac{I_1[b\sqrt{(t-\tau)^2-r^2/c^2}]}{\sqrt{(t-\tau)^2-r^2/c^2}}\,d\tau dS, \quad (9)$$

$$\varphi_2(t, M_0)=-\iint_\Gamma \cos(\overrightarrow{MM_0}, n)\,E\left(t-\frac{r}{c}\right)\exp(-rc/2a)\times$$

$$\times\left\{\frac{\psi_2(t-r/c, M)}{4\pi r^2}+\frac{1}{4\pi rc}\frac{\partial\psi_2(t-r/c, M)}{\partial t}\right\}dS+\frac{b}{4\pi c^3}\iint_\Gamma r\cos(\overrightarrow{MM_0}, n)\,E\left(t-\frac{r}{c}\right)\int_0^{t-r/c}\psi_2(\tau, M)\times$$

$$\times\exp[-c^2(t-\tau)/2a]\left\{b\,\frac{I_0[b\sqrt{(t-\tau)^2-r^2/c^2}]}{(t-\tau)^2-r^2/c^2}-2\,\frac{I_1[b\sqrt{(t-\tau)^2-r^2/c^2}]}{[\sqrt{(t-\tau)^2-r^2/c^2}]^3}\right\}d\tau dS. \quad (10)$$

The functions φ_1 and φ_2 obtained in this way satisfy Eq. (1) and the initial conditions (2) and are smooth in domain D. In passing through the surface Γ the function $\varphi_1(t, M)$ remains continuous, but the functions $\varphi_2(t, M)$ and $\partial\varphi_1(t, M)/\partial n$ have discontinuities whose magnitudes can be determined by taking the inverse Laplace transforms of Eqs. (4a) and (5a)

$$\frac{\partial\varphi_1^{\mp}(t, M_0)}{\partial n_0}=\mp\frac{\psi_1(t, M_0)}{2}+\frac{\partial\varphi_1(t, M_0)}{\partial n_0},\quad M_0\in\Gamma, \quad (9a)$$

$$\varphi_2^{\mp}(t, M_0)=\pm\frac{\psi_2(t, M_0)}{2}+\varphi_2(t, M_0),\quad M_0\in\Gamma, \quad (10a)$$

where the plus and minus signs have their previous meanings. We call the functions φ_1 and φ_2 the single- and double-layer potentials for the telegraphers' equation (1). The volume potential in the three-dimensional case has the form

$$\varphi_3(t, M_0)=\iiint \frac{\psi_3(t-r/c, M)}{4\pi r}E\left(t-\frac{r}{c}\right)\exp(-rc/2a)\,dV+$$

$$+\frac{b}{4\pi c}\iiint E\left(t-\frac{r}{c}\right)\int_0^{t-r/c}\psi_3(\tau, M)\exp[-c^2(t-\tau)/2a]\times$$

$$\times I_1[b\sqrt{(t-\tau)^2-r^2/c^2}]/\sqrt{(t-\tau)^2-r^2/c^2}\,d\tau dV. \quad (11)$$

Within the domain of integration the volume potential (11) satisfies the inhomogeneous telegraphers' equation

$$\Delta\varphi_3+d\varphi_3-\frac{1}{c^2}\frac{\partial^2\varphi_3}{\partial t^2}-\frac{1}{a}\frac{\partial\varphi_3}{\partial t}=-\psi_3(t, M_0), \quad (12)$$

and outside the domain of integration it satisfies the homogeneous equation (1).

The potentials introduced consist of two terms, one of which describes an undistorted wave propagation process with damping, and the second a distorted diffuse track which remains in the medium when the wave passes through. In the limit $a\to\infty$, $d\to 0$ the potentials (9)-(11) are transformed into corresponding wave potentials [3]. In the other limiting case $c\to\infty$, $d\to 0$, Eqs. (9)-(11) are transformed into heat potentials [3]. To prove this it is necessary to use the leading terms in the asymptotic expansions of I_0 and I_1 and the inequality $(t-\tau)\gg r/c$ as $c\to\infty$. The potentials (9)-(11) can be used to solve various kinds of heat-conduction problems for the hyperbolic equation. The problem for the telegraphers' equation (1) with nonzero initial conditions can be reduced in the usual way to a problem with zero initial conditions (2) [3, 6].

Method of Integral Equations for Internal Boundary-Value Heat-Conduction Problems. In accord with the Vernotte–Lykov hypothesis we assume that the heat-conduction process is described by a hyperbolic equation with constant coefficients and a heat source Q which depends on coordinates and time. In generalized variables $Fo = at/l^2$, $R = r/l$, it has the form [7, 8]

$$-\gamma^2 \frac{\partial^2 u}{\partial Fo^2} - \frac{\partial u}{\partial Fo} + \Delta u + Q^* = 0, \; M_0 \in D, \tag{13}$$

$$Q^*(Fo, M_0) = \frac{l^2}{\lambda} \left[Q(Fo, M_0) + \gamma^2 \frac{\partial Q(Fo, M_0)}{\partial Fo} \right]. \tag{13a}$$

Here the dimensionless quantity $\gamma = a/lc$ is the reciprocal of the velocity of propagation of heat in the body. The heat flux $q(Fo, M_0)$ in domain D is related to the temperature by the expression [7, 8]

$$\gamma^2 \frac{\partial q}{\partial Fo} + q = -\frac{\lambda}{l} \nabla u. \tag{14}$$

We assume that zero initial conditions are satisfied in domain D

$$u(0, M_0) = \partial u(0, M_0)/\partial Fo = 0. \tag{15}$$

In formulating boundary-value heat-conduction problems for domain D with boundary Γ, one of the following three boundary conditions must be specified:

$$u(Fo, M_0) = u_0(Fo, M_0), \; M_0 \in \Gamma, \tag{16}$$

$$q(Fo, M_0) = q_0(Fo, M_0), \; M_0 \in \Gamma, \tag{17}$$

$$q(Fo, M_0) = \alpha [u_0(Fo, M_0) - u(Fo, M_0)], \; M_0 \in \Gamma. \tag{18}$$

Here u_0 is the temperature of the medium, q_0 is the external heat source, and α is the constant heat-transfer coefficient at the boundary Γ.

We introduce the operators K_{10} and K_{20} corresponding to the wave process of heat propagation with damping:

$$\begin{gathered} K_{10}\psi = l \iint_\Gamma \frac{\psi(Fo - \gamma R, M)}{4\pi R} E(Fo - \gamma R) \exp(-R/2\gamma)\, dS, \\ K_{20}\psi = \iint_\Gamma \cos(\overrightarrow{MM_0}, n)\, E(Fo - \gamma R) \exp(-R/2\gamma) \times \\ \times \left\{ \frac{\psi(Fo - \gamma R, M)}{4\pi R^2} + \frac{\gamma}{4\pi R} \frac{\partial \psi(Fo - \gamma R, M)}{\partial Fo} \right\} dS, \end{gathered} \tag{19}$$

and the operators K_{11} and K_{21} which describe a diffuse track in the medium

$$\begin{gathered} K_{11}\psi = \frac{l}{8\pi\gamma} \iint_\Gamma E(Fo - \gamma R) \int_0^{Fo} \psi(Fo', M)\, E(Fo - Fo' - \gamma R) \times \\ \times \exp[-(Fo - Fo')/2\gamma^2] \frac{I_1\left[\frac{1}{2\gamma^2} \sqrt{(Fo - Fo')^2 - \gamma^2 R^2}\right]}{\sqrt{(Fo - Fo')^2 - \gamma^2 R^2}} \, dFo'\, dS, \\ K_{21}\psi = \frac{1}{8\pi} \iint_\Gamma R \cos(\overrightarrow{MM_0}, n)\, E(Fo - \gamma R) \int_0^{Fo} \psi(Fo', M) \times \\ \times E(Fo - Fo' - \gamma R) \exp[-(Fo - Fo')/2\gamma^2] \left\{ \frac{1}{2\gamma} I_0\left[\frac{1}{2\gamma^2} \sqrt{(Fo - Fo')^2 - \gamma^2 R^2}\right] - \right. \\ \left. - 2\gamma \frac{I_1\left[\frac{1}{2\gamma^2} \sqrt{(Fo - Fo')^2 - \gamma^2 R^2}\right]}{\sqrt{(Fo - Fo')^2 - \gamma^2 R^2}} \right\} \frac{dFo'\, dS}{(Fo - Fo')^2 - \gamma^2 R^2}. \end{gathered} \tag{20}$$

In terms of these operators the single- and double-layer potentials are

$$\varphi_1(\mathrm{Fo},\ M_0) = K_{10}\psi_1 + K_{11}\psi_1, \tag{21}$$

$$\varphi_2(\mathrm{Fo},\ M_0) = K_{20}\psi_2 + K_{21}\psi_2. \tag{22}$$

In generalized variables the volume potential (11) for Eq. (13) takes the form

$$\varphi_3(\mathrm{Fo},\ M_0) = l^2 \iiint_D \frac{Q^*(\mathrm{Fo}-\gamma R,\ M)}{4\pi R} E(\mathrm{Fo}-\gamma R)\exp(-R/2\gamma)\,dV +$$

$$+\frac{l^2}{8\pi\gamma}\iiint_D E(\mathrm{Fo}-\gamma R)\int_0^{\mathrm{Fo}} Q^*(\mathrm{Fo}',\ M)\,E(\mathrm{Fo}-\mathrm{Fo}'-\gamma R)\times$$

$$\times\exp[-(\mathrm{Fo}-\mathrm{Fo}')/2\gamma^2]\ \frac{I_1\left[\frac{1}{2\gamma^2}\sqrt{(\mathrm{Fo}-\mathrm{Fo}')^2-\gamma^2R^2}\right]}{\sqrt{(\mathrm{Fo}-\mathrm{Fo}')^2-\gamma^2R^2}}\,d\,\mathrm{Fo}'\,dV. \tag{23}$$

We seek the solution of the first boundary-value problem as a sum of double-layer and volume potentials, i.e., $u = \varphi_2(\mathrm{Fo},\ M_0) + \varphi_3(\mathrm{Fo},\ M_0)$. Taking account of (10a), boundary condition (16) leads to the following equation for $\psi_2(\mathrm{Fo},\ M_0)$:

$$\psi_2(\mathrm{Fo},\ M_0) + 2K_{20}\psi_2 + 2K_{21}\psi_2 = 2u_0(\mathrm{Fo},\ M_0) - 2\varphi_3(\mathrm{Fo},\ M_0),\ M_0\in\Gamma. \tag{24}$$

We seek the solution of the second boundary-value problem as a sum of single-layer and volume potentials, i.e., $u = \varphi_1(\mathrm{Fo},\ M_0) + \varphi_3(\mathrm{Fo},\ M_0)$. The boundary condition (17) can be replaced by the equivalent but more convenient form

$$\frac{\lambda}{l}\ \frac{\partial u(\mathrm{Fo},\ M_0)}{\partial n_0} = \gamma^2\ \frac{\partial q_0}{\partial\,\mathrm{Fo}} + q_0,\ M_0\in\Gamma. \tag{17a}$$

Taking account of Eq. (9a), boundary condition (17a) leads to an equation of the second kind for $\psi_1(\mathrm{Fo},\ M_0)$:

$$\psi_1(\mathrm{Fo},\ M_0) - 2\tilde{K}_{20}\psi_2 - 2\tilde{K}_{21}\psi_2 = -2\ \frac{q_0}{\lambda} - 2\gamma^2\ \frac{\partial q_0}{\partial\,\mathrm{Fo}} + \frac{2}{l}\ \frac{\partial\varphi_3}{\partial n_0},\ M_0\in\Gamma. \tag{25}$$

In Eq. (25) the operators $\tilde{K}_{20}$ and $\tilde{K}_{21}$ differ from the operators K_{20} and K_{21} only in that the angle $(\overrightarrow{MM_0},\ n)$ between the vector $\overrightarrow{MM_0}$ and the inward normal n to the boundary Γ at the point M is replaced by the angle $(\overrightarrow{M_0M},\ n_0)$ between the vector $\overrightarrow{M_0M}$ from point M_0 to point M and the inward normal n_0 at point M_0 of the boundary Γ.

We seek the solution of the third boundary value problem as the sum of single-layer and volume potentials, i.e., $u = \varphi_1(\mathrm{Fo},\ M_0) + \varphi_3(\mathrm{Fo},\ M_0)$. By differentiating Eq. (18) with respect to Fo and using (14) we reduce the third boundary condition of (18) to the more convenient form

$$\frac{\partial u(\mathrm{Fo},\ M_0)}{\partial n_0} - \mathrm{Bi}\,[u_0(\mathrm{Fo},\ M_0) - u(\mathrm{Fo},\ M_0)] - \gamma^2\,\mathrm{Bi}\left[\frac{\partial u_0(\mathrm{Fo},\ M_0)}{\partial\,\mathrm{Fo}} - \frac{\partial u(\mathrm{Fo},\ M_0)}{\partial\,\mathrm{Fo}}\right] = 0,\ M_0\in\Gamma. \tag{26}$$

When (9a) is taken into account, (26) leads to the following equation for ψ_1:

$$\psi_1(\mathrm{Fo},\ M_0) - 2l\,[\tilde{K}_{20}+\tilde{K}_{21}]\,\psi_1 - 2\,\mathrm{Bi}\,[K_{10}+K_{11}]\,\psi_1 -$$

$$-2\gamma^2\,\mathrm{Bi}\ \frac{\partial}{\partial\,\mathrm{Fo}}\ [K_{10}\psi_1 + K_{11}\psi_1] = 2\,\mathrm{Bi}\,[\varphi_3(\mathrm{Fo},\ M_0) - u_0(\mathrm{Fo},\ M_0)] +$$

$$+2\gamma^2\,\mathrm{Bi}\left[\frac{\partial\varphi_3(\mathrm{Fo},\ M_0)}{\partial\,\mathrm{Fo}} - \frac{\partial u_0(\mathrm{Fo},\ M_0)}{\partial\,\mathrm{Fo}}\right] + 2\ \frac{\partial\varphi_3(\mathrm{Fo},\ M_0)}{\partial n_0},\ M_0\in\Gamma. \tag{27}$$

Equations (24), (25), and (27) are the same type of integrodifferential equations of the second kind for the unknown function in which the derivative of the unknown function enters under the integral sign. They can be solved by methods usually applicable to integral equations, such as iteration, expansion in terms of a small parameter, etc. The process of the gradual propagation of a thermal field in space and time and the process of attenuation of a propagating wave appreciably simplify the numerical solution of Eqs. (24), (25), and (27).

The method of potentials described is easily extended to domains having several boundaries. In addition it can be used to solve external boundary-value problems and linear inverse boundary-value heat-conduction problems. Let us apply the potential method to solve the last problem, which has the following mathematical formulation. In domain D_1 ($D \subset D_1$) with boundary Γ_1 find the solution of Eq. (13) with the initial conditions (15) in which temperature (or heat flux) transducers are placed on the boundary Γ of domain D, which is a subdomain of D_1. In this case one of the following conditions of measurement is satisfied:

$$u(\mathrm{Fo},\ M_0) = u_0(\mathrm{Fo},\ M_0),\ M_0 \in \Gamma, \tag{28a}$$

$$q(\mathrm{Fo},\ M_0) = q_0(\mathrm{Fo},\ M_0),\ M_0 \in \Gamma. \tag{28b}$$

These conditions can be satisfied in practice by measuring the temperature (or heat flux) at a finite number of points on the surface Γ and using some interpolation process to construct the function $u_0(\mathrm{Fo}, M_0)$ or $q_0(\mathrm{Fo}, M_0)$ over the whole surface. This function will describe the true temperature distribution (or heat flux) on Γ approximately. By solving the direct boundary value problem in domain D with boundary condition (28a) or (28b) we determine the temperature u and the derivative $\partial u/\partial n$ in domain D and on its boundary. We find the temperature u in domain $D_1 - D$ which satisfies Eq. (13) and the initial conditions (15) by solving the problem of continuing the temperature distribution into the exterior of domain D by means of the temperature and the derivative $\partial u/\partial n$ specified on the boundary Γ of domain D

$$\begin{aligned} &u(\mathrm{Fo},\ M_0) = u_0(\mathrm{Fo},\ M_0),\ M_0 \in \Gamma, \\ &\partial u(\mathrm{Fo},\ M_0)/\partial n_0 = u_1(\mathrm{Fo},\ M_0),\ M_0 \in \Gamma. \end{aligned} \tag{29}$$

We seek the solution of this problem as the sum of the volume potential in domain $D_1 - D$ and two single-layer potentials. Instantaneous point sources of the first of these are placed on the surface Γ of domain D with a density $\psi_1(\mathrm{Fo}, M)$, and sources of the second are placed on the surface Γ_1 with a density $\psi_2(\mathrm{Fo}, M)$. Denoting the operators K_{10}, K_{11}, K_{20}, and K_{21} for the surface Γ_1 by $K_{10}^{(2)}$, $K_{11}^{(2)}$, $K_{20}^{(2)}$, and $K_{21}^{(2)}$, we have

$$u(\mathrm{Fo},\ M_0) = \varphi_3(\mathrm{Fo},\ M_0) + (K_{10} + K_{11})\psi_1 + (K_{10}^{(2)} + K_{11}^{(2)})\psi_2. \tag{30}$$

By using (9a) we satisfy boundary conditions (29). As a result we obtain a single integro-differential equation for the two unknown functions ψ_1 and ψ_2 which is of the type considered above

$$\psi_1(\mathrm{Fo},\ M_0) + 2\tilde{K}_{20}\psi_1 + 2\tilde{K}_{21}\psi_1 + 2\tilde{K}_{20}^{(2)}\psi_2 + 2\tilde{K}_{21}^{(2)}\psi_2 = u_1(\mathrm{Fo},\ M_0) - \frac{\partial\varphi_3(\mathrm{Fo},\ M_0)}{\partial n_0};\ M_0 \in \Gamma \tag{31}$$

and one integral relation

$$u_0(\mathrm{Fo},\ M_0) = \varphi_3(\mathrm{Fo},\ M_0) + (K_{10} + K_{11})\psi_1 + (K_{10}^{(2)} + K_{11}^{(2)})\psi_2,\ M_0 \in \Gamma. \tag{32}$$

To solve a linear inverse boundary-value heat-conduction problem it is necessary to know the value of γ, which corresponds to the existing velocity of propagation of heat in the body. It can be determined experimentally [1]. Equations (31) and (32) can be solved by the methods used in solving (24), (25), and (27).

By integrating the potentials introduced once or twice with respect to coordinates, two-dimensional or one-dimensional potentials can be obtained for the telegraphers' equation (1). In addition, the potentials introduced can be generalized to the case of several domains, permitting the solution of direct and inverse heat-conduction problems for such domains. The potentials for Eq. (1) can also be used to solve other inverse heat-conduction problems.

NOTATION

λ, thermal conductivity; a, thermal diffusivity; c, velocity of propagation of heat; p, Laplace variable; $\delta(t)$, Dirac delta function; I_0, I_1, Bessel functions of the first kind of imaginary argument; E(z), Heaviside unit function; l, characteristic dimension of body; Bi, Biot number; Fo, Fourier number; γ, dimensionless quantity, reciprocal of velocity of propagation of heat.

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STEADY-STATE HEAT CONDUCTION IN COMPOSITE SYSTEMS WITH BOUNDARY CONDITIONS OF THE FOURTH KIND

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A method is proposed for solving steady-state heat-conduction problems in a system of contacting regions, and examples are presented to illustrate its effectiveness.

As high-temperature thermal physics develops, problems of determining temperature distributions in systems of contacting bodies become more and more important [1]. We present a method for solving steady-state heat-conduction problems with matching boundary conditions based on the application of well-developed methods of solving elliptic differential equations in regions with piecewise homogeneous media [2, 3] and mathematical optimization methods (Hooke, Rosenbrock) [4].

The method involves the following steps.

1. A mathematical statement of the problem of determining the temperature distribution in a system of N contacting bodies is formulated.

2. Boundary conditions of the fourth kind on interfaces S_{ij} between the i-th and j-th regions of the original problem are replaced by boundary conditions of the second kind

$$\lambda_i \left.\frac{\partial U_i}{\partial n}\right|_{S_{ij}} = q_i(S_{ij}),$$
$$\lambda_j \left.\frac{\partial U_j}{\partial n}\right|_{S_{ij}} = q_j(S_{ij}), \quad (1)$$
$$q_i(S_{ij}) = -q_j(S_{ij}),$$

where $q_i(S_{ij})$ and $q_j(S_{ij})$ are unknown heat flux distribution functions on the boundary S_{ij} between the i-th and j-th regions. In this way the original boundary-value problem for determining the temperature distribution in the system is separated into N independent problems.

3. It is assumed that the functions $q_i(S_{ij})$ can be expressed by polynomials or step function representations by using one of the known methods of constructing a solution in each region. The solutions obtained in this way are parametrically dependent on the coefficients Q_{ik} which appear in the functions $q_i(S_{ij})$ and also on the constants C_i for an internal Neumann problem.

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